

Understanding Classical Stability

Jordi Gaset Rifà

in collaboration with Verónica Errasti Díez, Manuel Lainz and Georgina Staudt

Higher Time-Derivative Theories Online Seminar Series

October 20, 2026

Notions of stability

Who is stable?

- **Global stability.** All solutions satisfy the stability property.
- **Islands of stability.** Some solutions satisfy the stability property.
- **Local stability.** Solutions around a critical point or another solution satisfy the stability property.

For how long are they stable?

1. Solutions are bounded. Lagrange stability.
2. Solutions are either bounded or diverge only as time tends to infinity (runaway). This excludes solutions that diverge within finite time (blow ups). Bening ghosts.
3. Blow-ups only occur at long enough time. “Long-lived”.

When considering global stability, these will be called **G1**, **G2** and **G3** respectively.

Related areas in Mathematics

- **Lyapunov stability:** Local stability, infinite future time. Considers both 1 (Lagrange stability) and 2 (benign ghosts). A lot of applications. Very developed, including infinite dimensions.
- **Integrability:** Techniques very close to G1 and G2, as we will see. Usually, compactness of level sets is assumed, not proven.
- **Symplectic geometry:** Islands of stability, bounded solutions. Have some techniques to prove compactness of infinite dimensional sets.
- **KAM:** Islands of stability and local stability. Interested in long but finite time. Nekhoroshev estimates, Arnold diffusion.
- **Riemannian geometry:** G2 correspond to complete vector fields. Proving geodesic completeness of pseudo-Riemannian manifolds is similar to proving benign ghost.
- **PDE/Classical Field Theories:** Future slide.

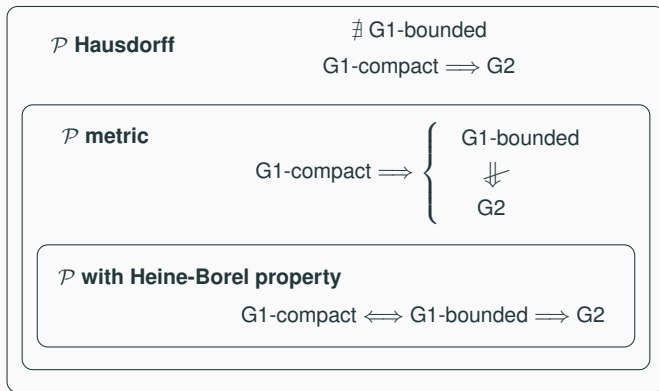
Formal definitions

Let $\mathcal{P}$ and N be two topological spaces. Usually, $N = \mathbb{R}$.

$$\text{Sol} = \{\gamma : I_\gamma \subset N \rightarrow \mathcal{P} \text{ continuous}\},$$

- For $\mathcal{P}$ metric, a system is **G1 in the bounded sense** if, for all curves $\gamma \in \text{Sol}$, their image $\gamma(I_\gamma)$ is bounded.
- For $\mathcal{P}$ Hausdorff, a system is **G1 in the compact sense** if, for all curves $\gamma \in \text{Sol}$, their image $\gamma(I_\gamma)$ is relatively compact with respect to $\mathcal{P}$ (inside a compact set of $\mathcal{P}$).
- For $\mathcal{P}$ Hausdorff, a system is **G2** if, for all curves $\gamma \in \text{Sol}$, and for all subsets $S \subset I_\gamma$ relatively compact with respect to $\mathbb{R}$, the image $\gamma(S)$ is relatively compact with respect to $\mathcal{P}$.
($\gamma(t) = v_0 t$, $I_\gamma = \mathbb{R}$).

Relations between global stability notions



All metric spaces are Hausdorff. All relatively compact sets are bounded, the converse is not true. $\mathcal{P}$ has the **Heine-Borel property** if all bounded sets are relatively compact. Examples of spaces without this property: infinite-dimensional Banach and Hilbert spaces, geodesically incomplete Riemannian manifolds $(\mathbb{R}^n / \{0\})$.

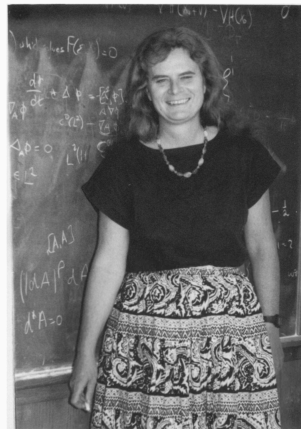
Yang-Mills equation

$$E_{YM} = \int_M |F_A|^2 dV_g.$$

For the Euclidean case:

- There exists solutions with constant energy but a blow-up of the curvature at finite points (not bounded in $\|\cdot\|_\infty$ norm). This is called bubbling.
- Uhlenbeck singularity theorem: the singular points can be regularized smoothly.
- Uhlenbeck compactness theorem: $\|\cdot\|_p$ -bounded set of solutions are compact.

Similar results hold for the non-euclidean case. It is an active field.



Uhlenbeck (1982). "Connections with L^p bounds on curvature". "Removable singularities in Yang-Mills fields". Comm. Math. Phys. Krieger, Schlag, Tataru. arXiv:math/0610248 [math.AP]

Heat equation with potential

The stability of the heat equations with potential has been studied intensively

$$u_t - \Delta u = f(t, x^i, u, \nabla u)$$

It uses Lyapunov methods base on the Energy (for $f(t, x^i, u)$ and $F' = f$):

$$\int_{\Sigma} \frac{1}{2} \|\nabla u\|_2^2 - F dx.$$

It is not conserved but decreases on solutions. There are examples of G1, G2, islands of stability and complete instability (future time). For some stable cases, the Energy is not bounded (sublinear case, for instance).

Let $u : [0, T] \times \Sigma \rightarrow S^2$ and $f = |\nabla u|^2 u$.

Given a set of points of Ω there exists initial and boundary conditions such that the solution has finite time blow-up at the specified points. They can be regularized, but discontinuities appear at the singular points.

Galaktionov, Mitidieri, Pohozaev (2014). "Blow-up for Higher-Order Parabolic, Hyperbolic, Dispersion and Schrodinger Equations".

Davila, del Pino, Juncheng Wei, arXiv:1702.05801 [math.AP]

Fluid equations

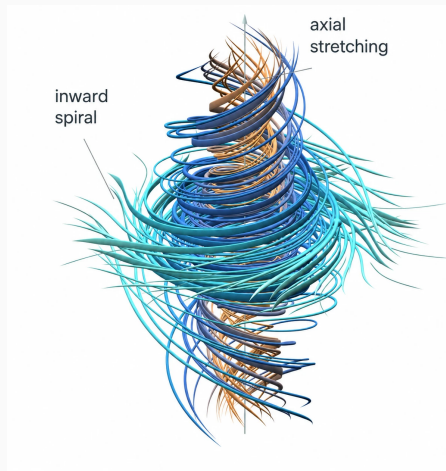
Millennium problem: Navier–Stokes existence and smoothness.
Simplifying: prove that it does not blow-up for any smooth initial conditions.

The blow-up with respect Sobolev norms is the sign of turbulence (Jean Leray, 1935). The (m, p) Sobolev norm is the sum of the L^p norms of the function and its derivatives up to order m .

If the vorticity $w(t)$ is bounded for all T :

$$\int_0^T \|\omega(t)\|_{L^\infty} dt < k(T)$$

then the solutions do not have blow-ups in the Sobolev norms.
That is, they do not have turbulence.



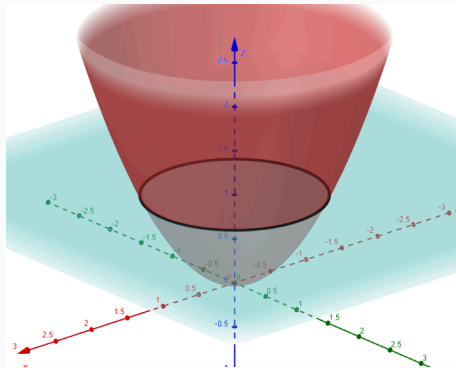
Lessons

- Examples of stable and unstable field theories exist (See also the works of Dmitry Kaparulin and collaborators, as well as Andrei Smilga and collaborators).
- Choosing the norm is very relevant.
- Not all blow-ups are unphysical. Blow-ups of $\|\cdot\|_\infty$ can be regularized and they signal the presence of instantons or discontinuities. Blow-ups of Sobolev norms signal turbulence.
- There are other methods to prove stability that do not require a conserved quantity. Lyapunov functions, bounded vorticity.
- We are lacking a mechanism, an understanding. Sobolev norms provide a mechanism to understand turbulence. Can we do the same for stable higher-order field theories? (Aaron Held's talk).

Why the energy works?

Theorem Given a system Sol with $\mathcal{P} = \mathbb{R}^n$, if the energy $E : \mathbb{R}^n \rightarrow \mathbb{R}$ is a twice differentiable conserved quantity that has a minimum and whose Hessian is positive definite everywhere, then the system is G1.

Idea of the proof: E has bounded level sets and the solutions are inside level sets.



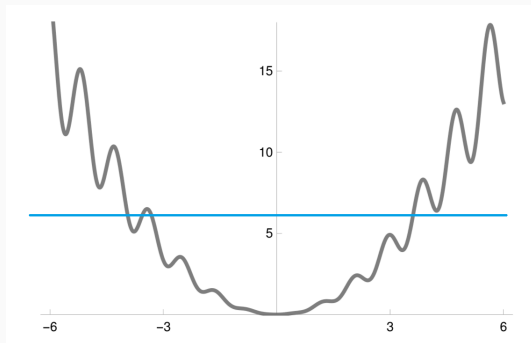
An improvement: coercive conserved quantities

Definition¹ A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is **coercive** if the absolute value of $f(x)$ diverges as the modulus of x diverges:

$$\lim_{\|x\| \rightarrow +\infty} |f(x)| \rightarrow +\infty.$$

Proposition If f is coercive, it has bounded level sets.

Theorem Given a system Sol with $\mathcal{P} = \mathbb{R}^n$, if there exists a conserved quantity $Q : \mathbb{R}^n \rightarrow \mathbb{R}$ that is coercive, then the system is G1.



¹ It can be extended to $f : M \rightarrow N$, with M and N a metric spaces and M has the Heine-Borel property.

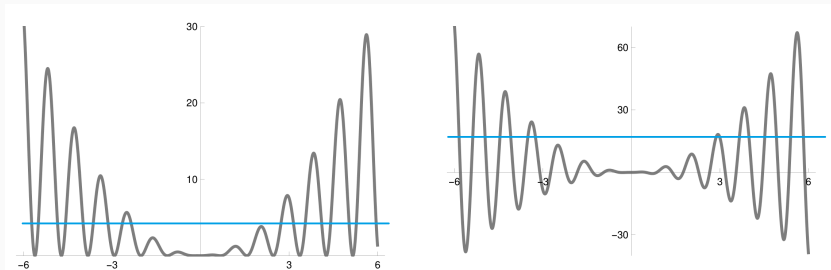
Deffayet, Held, Mukohyama, Vikman, arXiv:2305.09631 [gr-qc], arXiv:2108.06294 [gr-qc]. Heredia, Llosa, arXiv:2403.19777 [hep-th] .

A further improvement: confining conserved quantities

Definition Let $\mathcal{P}$ be a Hausdorff space and let $\mathcal{S}$ be a set. A function $f : \mathcal{P} \rightarrow \mathcal{S}$ is **confining** if the path-connected components of its level sets are relatively compact with respect to $\mathcal{P}$.

$$\{f \text{ with def. pos. Hessian and minimum}\} \subset \{f \text{ coercive}\} \subset \{f \text{ confining}\}$$

Theorem Given a system Sol with $\mathcal{P}$ Hausdorff and N a topological space. If there exists a conserved quantity $Q : \mathcal{P} \rightarrow N$ that is confining, then the system is G1 in the compact sense.



Example

$$L = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) - \frac{\omega^2}{2}(x^2 - y^2) - f(x + \alpha y),$$

$$E_L = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) + \frac{\omega^2}{2}(x^2 - y^2) + f(x + \alpha y)$$

It has the conserved quantity:

$$Q = \frac{1}{2} \left(\dot{x}^2 + \dot{y}^2 + \frac{4\alpha}{1 + \alpha^2} \dot{x}\dot{y} \right) + \frac{\omega^2}{2} \left(x^2 + y^2 + \frac{4\alpha}{1 + \alpha^2} xy \right) + \frac{1 - \alpha^2}{1 + \alpha^2} f(x + \alpha y).$$

- The symmetry associated to Q is not natural.
- It is bi-Hamiltonian for $\alpha \neq \pm 1$. ($E_L, \omega_L = dx \wedge d\dot{x} - dy \wedge d\dot{y}$) and

$$(Q, \hat{\omega} = dx \wedge d\dot{x} + c_1 dx \wedge d\dot{y} + dy \wedge d\dot{y} + c_1 dy \wedge d\dot{x})$$

generate the same dynamics and ω_L and $\hat{\omega}$ are compatible.

Case in point

$$f(x + \alpha y) = b(x + \alpha y)^2 \sin(x + \alpha y).$$

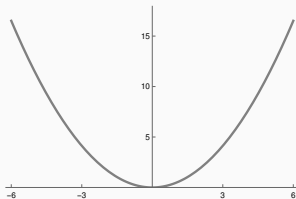
a) $b = 0$, the system decouples. Q has positive definite Hessian everywhere and minimum.

b) $|b| \in (0, b_c)$, Q is coercive.

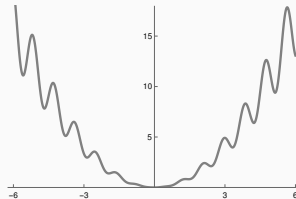
c) $|b| = b_c = \frac{\omega^2}{2|1-\alpha^2|}$ Q is confining with (infinitely many) global minima.

d) $|b| \in (b_c, \infty)$, Q is confining and unbounded from above and below.

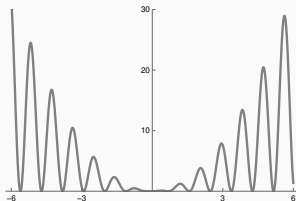
A) Hessian pos-def, bounded from below



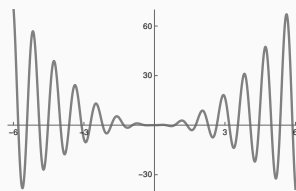
B) Coercive, bounded from below – Hessian not pos-def



C) Confining, bounded from below – Non-coercive



D) Confining, totally unbounded – Non-coercive



Multiple Conserved Quantities

Let $Q_i : M \rightarrow N$, for $i = 1, \dots, r$ conserved quantities of a system. Define

$$F = (Q_1, \dots, Q_r) : M \rightarrow N^n$$

The level sets of F are the intersections of the level sets of the Q_i .

Lemma Let M be a Hausdorff space and let W and V be two sets. If $F : M \rightarrow W$ is not confining, then for any function $g : W \rightarrow V$, $g \circ F$ is not confining.

Namely, if F is not confining no functional combinations of the conserved quantities will be confining.

Global Stability for Integrable Systems

Consider a Hamiltonian system (ω, H) with n conserved quantities $\{Q_i\}_{i=1,\dots,n}$ in involution $\{Q_i, Q_j\} = 0$ such that dQ_i are generically independent. The corresponding Hamiltonian vector fields, or symmetries, are the unique vector fields X_{Q_i} satisfying

$$i_{X_{Q_i}}\omega = dQ_i.$$

Notice that such a system is not necessarily integrable because the symmetries may not be complete. Define

$$F = (Q_1, \dots, Q_n) : M \rightarrow \mathbb{R}^n$$

First Case: F is confining.

The connected components of generic level sets of F are tori and action-angle variables are defined on a neighbourhood of these tori. Thus, the integral curves are trapped inside **tori** and the system is **G1 stable** on the tori.

Global Stability for Integrable Systems

Second Case: F is not confining but the vector fields $\{X_{Q_i}\}_{i=1,\dots,n}$ generate a transitive action on the connected components of the level sets of F .

The connected components of the level sets of regular points are cylinders $\mathbb{T}^k \times \mathbb{R}^{n-k}$.

Global Stability for Integrable Systems

Second Case: F is not confining but the vector fields $\{X_{Q_i}\}_{i=1,\dots,n}$ generate a transitive action on the connected components of the level sets of F .

The connected components of the level sets of regular points are cylinders $\mathbb{T}^k \times \mathbb{R}^{n-k}$.

On a neighbourhood of each cylinder there exist action-angle coordinates $(\theta^a, s^A, I_a, \bar{I}_A)$.

θ^a are position coordinates on $\mathbb{T}^k$, s^A are position coordinates on $\mathbb{R}^{n-k}$ and $(I_a, \bar{I}_A)$ are the respective momenta. The equations of motion are

$$\begin{aligned}\dot{\theta}^a &= \frac{\partial H(I_a, \bar{I}_A)}{\partial I_a} =: \omega_a(I_a, \bar{I}_A), & \dot{I}_a &= 0, \\ \dot{s}^A &= \frac{\partial H(I_a, \bar{I}_A)}{\partial \bar{I}_A} =: \bar{\omega}_A(I_a, \bar{I}_A), & \dot{\bar{I}}_A &= 0.\end{aligned}$$

The existence of these coordinates guarantees that the system is $G2$ stable on the cylinder.

Global Stability for Integrable Systems

Second Case:

The equations of motion are

$$\begin{aligned}\dot{\theta}^a &= \frac{\partial H(I_a, \bar{I}_A)}{\partial I_a} =: \omega_a(I_a, \bar{I}_A), & \dot{I}_a &= 0, \\ \dot{S}^A &= \frac{\partial H(I_a, \bar{I}_A)}{\partial \bar{I}_A} =: \bar{\omega}_A(I_a, \bar{I}_A), & \dot{\bar{I}}_A &= 0.\end{aligned}$$

- If $\bar{\omega}_A(I_a, \bar{I}_A)$ is not identically zero on some cylinder. The system is $G2$ stable on the cylinders but not $G1$ stable.
- If $\bar{\omega}_A(I_a, \bar{I}_A)$ is identically zero for all A and every cylinder. The system is $G1$ stable on the cylinders. Furthermore, the following map is confining:

$$(Q_i, \dots, Q_n, s^{k+1}, \dots, s^n) : M \rightarrow \mathbb{R}^{2n-k}$$

is confining because the connected components of its level sets are tori, which are compact.

Theorem Let (ω, H) be a Hamiltonian system with conserved quantities $\{Q_i\}_{i=1,\dots,n}$ in involution and independent everywhere. Assume that the Hamiltonian vector fields X_{Q_i} are complete on the common level sets of the Q_i 's. Then, the system is G1 stable if, and only if, there exists a smooth map $F : M \rightarrow \mathbb{R}^m$, with $m \geq n$, that is invariant by the dynamics and confining.

Conclusions

- Stability is mainly a topological notion.
- The confining conserved quantity property is tight (at least for integrable systems). The interest shifts to finding alternative methods to prove stability.
- There are in the literature alternative methods, based on Lyapunov functions and bounded quantities.
- What is the deal with bi-Hamiltonians?
- Determining the stability of field theories is very hard, and it is usually done case by case.
- The chosen metric is crucial in field theories. The blow-up of each metric has a different effect.
- Not all blow-ups are unphysical. They may be regularized, and could signal relevant phenomena.
- We are lacking a mechanism to explain stable higher-order theories.

Thank you for your attention!

jordi.gaset@cunef.edu